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CRACK AT THE APEX OF A LOADED NOTCH

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Abstract—This paper deals with the plane elastostatic problem of an infinite wedge, subjected to arbitrary surface tractions, and cracked along the wedge angle bisector. The problem is reduced to a single Fredholm integral equation, which is solved numerically for normal loads on the crack faces and various loads on the wedge faces. It is shown that the crack tip intensity factor depends strongly on the wedge angle. An approximation to a half plane with a notch of finite angle, cracked at its apex, is also obtained.

1. INTRODUCTION

In this paper, the problem of an infinite elastic wedge subjected to arbitrary surface tractions, with a crack along the wedge angle bisector, emanating from the wedge apex, is considered. The crack faces are subjected to normal loads. Of principal interest in this investigation is the stress intensity factor at the crack tip due to the applied loads.

The wedge geometry has been treated extensively in the literature. Srivastav and Narain[1] considered this same geometry with a crack on the wedge angle bisector, but no loads on the wedge faces. Muki and Westmann[2] considered a wedge with a crack propagating along the bond line emanating from the notch tip. However, they do not consider loading of the wedge faces. In a recent paper by Erdogan and Arin[3], the crackless wedge is treated for a class of contact problems in which the wedge is subjected to external loads through the forced contact with a rigid indenter of arbitrary profile.

In all of the above works, the solution was obtained through the use of Mellin transforms to reduce the boundary value problem to dual integral equations. A modification of the method used by Muki and Westmann is used here. The boundary conditions are represented in terms of the Mellin inverse transform of the 2-dimensional Airy stress function and a dual integral equation is written along the line of the crack. This dual integral equation is then reduced to a single Fredholm equation of the second kind, which is solved numerically.

The Fredholm integral equation is written for general loads on both the crack and wedge faces. The particular case of uniform normal loads is carried out in detail for various wedge angles and values for the stress intensity factor are obtained. By using the analytical results for an appropriately loaded infinite wedge, an approximation to a half plane with a cracked notch is obtained.

2. FORMULATION OF THE PROBLEM

This analysis considers an infinite elastic wedge occupying the region in the (r, θ) plane: $0 \le r < \infty$; $-\theta_0 \le \theta \le \theta_0$; $0 < \theta_0 < \pi$. The faces of the wedge are subjected to normal and shearing tractions, while the crack is loaded normally (see Fig. 1); due to the symmetry about $\theta = 0$, only the problem for $\theta > 0$ is considered. The boundary conditions are given below.

$$\tau_{\theta\theta} = N(r)H(a-r) \qquad \theta = \theta_0, \quad 0 \le r < \infty \tag{1}$$

$$\tau_{\theta r} = S(r)H(a-r) \qquad \theta = \theta_0, \quad 0 \le r < \infty \tag{2}$$

$$\tau_{\theta\theta} = -p(r) \qquad \theta = 0, \quad 0 \le r \le 1 \tag{3}$$

$$u_\theta \equiv v = 0 \qquad \theta = 0, \quad 1 \le r < \infty \tag{4}$$

$$\tau_{\theta r} = 0 \qquad \theta = 0, \quad 0 \le r < \infty \tag{5}$$

where $H(x)$ is the Heaviside function.

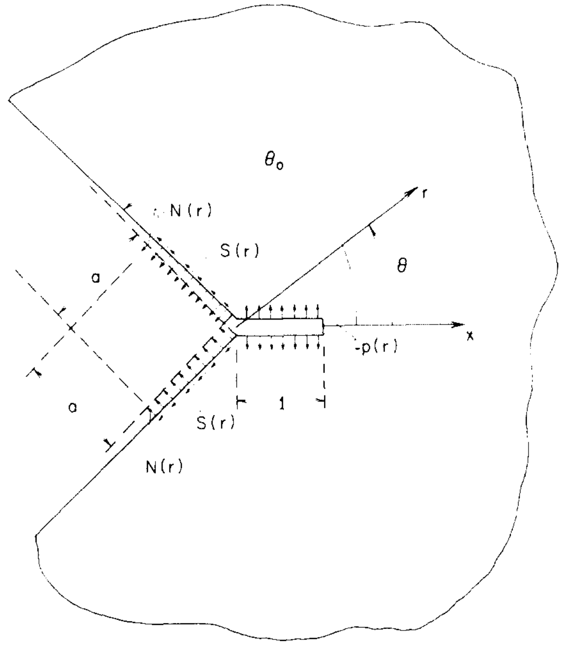

Fig. 1. Geometry and loading for an infinite wedge.

Using the Airy stress function in polar coordinates, the stresses and displacements can be written in terms of the Mellin transform of the stress function as follows:

$$r\tau_{rr} = \frac{1}{2\pi i}\int_{B_r}\left[\frac{\mathrm{d}^2\bar{\phi}}{\mathrm{d}\theta^2} + (1-s)\bar{\phi}\right]r^{-s}\,\mathrm{d}s \tag{6}$$

$$r\tau_{\theta r} = \frac{1}{2\pi i}\int_{B_r} s\frac{\mathrm{d}\bar{\phi}}{\mathrm{d}\theta}r^{-s}\,\mathrm{d}s \tag{7}$$

$$r\tau_{\theta\theta} = \frac{1}{2\pi i}\int_{B_r} s(s-1)\bar{\phi}r^{-s}\,\mathrm{d}s \tag{8}$$

$$2\mu u_r \equiv 2\mu u = \frac{1}{2\pi i}\int_{B_r}\left\{\tfrac{1}{4}[(3-\kappa)(s-1)^2 + 4(s-1)]\bar{\phi} + [\tfrac{1}{4}(3-\kappa) - 1]\frac{\mathrm{d}^2\bar{\phi}}{\mathrm{d}\theta^2}\right\}\frac{r^{-s}\,\mathrm{d}s}{s} \tag{9}$$

$$2\mu u_\theta \equiv 2\mu v = \frac{1}{2\pi i}\int_{B_r}\left\{\tfrac{1}{4}[4(2s^2 - s + 1) - (3-\kappa)(s-1)^2]\frac{\mathrm{d}\bar{\phi}}{\mathrm{d}\theta} + [1 - \tfrac{1}{4}(3-\kappa)]\frac{d^3\bar{\phi}}{\mathrm{d}\theta^3}\right\}\frac{r^{-s}\,\mathrm{d}s}{(s+1)s} \tag{10}$$

$$\bar{\phi}(s,\theta) \equiv \int_0^\infty \phi(r,\theta)r^{s-2}\,\mathrm{d}r \tag{11}$$

$$\kappa = 3 - 4\nu \qquad \text{plane strain} \tag{12a}$$

$$= (3-\nu)/(1+\nu) \qquad \text{plane stress} \tag{12b}$$

where ν is Poisson's ratio.

The nature of the geometric singularities occurring at the wedge apex has been thoroughly studied (see for instance Bogy[4]); hence, the following regularity conditions are prescribed for the stresses at critical points.

$$\tau_{rr}, \tau_{\theta\theta}, \tau_{\theta r} = 0(r^{-\alpha}) \qquad \alpha \in (-1, 1), r \to 0^+ \tag{13a}$$

$$\tau_{\theta\theta} = 0[(r-1)^{-1/2}] \qquad r \to 1^+ \tag{13b}$$

$$\tau_{rr}, \tau_{\theta\theta}, \tau_{\theta r} = 0(r^{-2}) \qquad r \to \infty. \tag{13c}$$

The behavior of the stresses at $r = 0^+$ and at $r = \infty$ determines the strip of regularity in which the

Bromwich path must lie in eqns (6)–(10). From eqns (13a, c) and the definition of the Mellin transform, eqn (11), it is found that

$$R(s) \in [\alpha - 1, 0) \tag{14}$$

in eqns (6)–(10).

The biharmonic equation for the stress function reduces to the elementary ordinary differential equation for the transformed stress function given below.

$$\left[\frac{d^2}{d\theta^2} + (s+1)^2\right]\left[\frac{d^2}{d\theta^2} + (s-1)^2\right]\bar{\phi}(s, \theta) = 0. \tag{15}$$

3. REDUCTION TO A DUAL INTEGRAL EQUATION

The following solution of eqn (15) is chosen such that the boundary conditions on the wedge face, eqns (1) and (2), are satisfied automatically.

$$\begin{aligned}\bar{\phi}(s, \theta) = {} & \bar{\phi}_w(s, \theta) + A(s)\sin(\pi s)\{(s+1)\sin[(s-1)(\theta-\theta_0)] \\ & -(s-1)\sin[(s+1)(\theta-\theta_0)]\} + B(s)\sin(\pi s) \\ & \times\{\cos[(s-1)(\theta-\theta_0)] - \cos[(s+1)(\theta-\theta_0)]\}\end{aligned} \tag{16a}$$

$$\begin{aligned}\bar{\phi}_w(s, \theta) = {} & \frac{1}{s\Delta(s, \theta_0)}\left\{\left[\bar{S}(s)\cos[(s+1)\theta_0] + \bar{N}(s)\frac{(s+1)}{(s-1)}\right.\right. \\ & \left.\times\sin[(s+1)\theta_0]\right]\cos[(s-1)\theta] - \left[\bar{S}(s)\cos[(s-1)\theta_0]\right. \\ & \left.\left.+\bar{N}(s)\sin[(s-1)\theta_0]\right]\cos[(s+1)\theta]\right\}\end{aligned} \tag{16b}$$

$$\Delta(s, \theta_0) = s\sin(2\theta_0) + \sin(2s\theta_0) \tag{16c}$$

$$\bar{N}(s) = \int_0^a N(r)r^s\,dr \tag{16d}$$

$$\bar{S}(s) = \int_0^a S(r)r^s\,dr, \tag{16e}$$

where $\bar{\phi}_w(s, \theta_0)$ is the solution to the crackless wedge with the loading given in eqns (1) and (2). This problem has been treated before (see for instance [4, 5]) for normal wedge loads and the solution is easily extended to both normal and shear loading.

Upon differentiating eqn (16a), substituting into eqn (7) and applying the shear stress boundary condition, eqn (5), one can solve for $A(s)$ in terms of $B(s)$,

$$A(s) = \frac{-B(s)}{(s+1)(s-1)}\frac{[(s-1)\sin[(s-1)\theta_0] - (s+1)\sin[(s+1)\theta_0]]}{\cos[(s-1)\theta_0] - \cos[(s+1)\theta_0]}. \tag{17}$$

Introduce the unknown,

$$D(s) = \frac{2B(s)\sin(\pi s)}{(s+1)\sin(s\theta_0)\sin(\theta_0)}\Delta(s, \theta_0) \tag{18}$$

and substitute eqn (18) first into eqn (17), then into eqn (16a) and differentiate appropriately for substitution into eqns (8) and (10). The displacement and stress boundary conditions on $\theta = 0$, eqns (3) and (4) respectively, can therefore be written as the dual integral equation in $D(s)$ given below.

$$-\frac{1}{2\pi i}\int_{B_r} D(s)r^{-s}\,ds = 0 \qquad 1 < r < \infty \tag{19}$$

$$-\frac{1}{2\pi i}\int_{B_r}\tfrac{1}{2}sD(s)\eta(s,\theta_0)r^{-s-1}\,ds$$

$$=\frac{1}{2\pi i}\int_{B_r}[\psi_N(s,\theta_0)\bar{N}(s)+\psi_S(s,\theta)\bar{S}(s)]r^{-s-1}\,ds+p(r),\quad 0<r<1 \tag{20a}$$

$$\eta(s,\theta_0)=\frac{s^2\cos(2\theta_0)-\cos(2s\theta_0)+1-s^2}{\Delta(s,\theta_0)} \tag{20b}$$

$$\psi_N(s,\theta_0)=\frac{-2[s\cos(s\theta_0)\sin(\theta_0)+\sin(s\theta_0)\cos(\theta_0)]}{\Delta(s,\theta_0)} \tag{20c}$$

$$\psi_S(s,\theta_0)=\frac{2(s-1)\sin(s\theta_0)\sin(\theta_0)}{\Delta(s,\theta_2)}. \tag{20d}$$

Noting that $\eta(s)\sim\tan(\pi s)$ as $|s|\to\infty$, eqn (20a) can be rewritten to obtain the dual integral equation,

$$\frac{1}{2\pi i}\int_{B_r}D(s)r^{-s}\,ds=0,\qquad 1<r<\infty \tag{21}$$

$$\frac{1}{2\pi i}\int_{B_r}\tfrac{1}{2}sD(s)\tan(\pi s)r^{-s-1}\,ds$$

$$=\frac{-1}{2\pi i}\int_{B_r}\tfrac{1}{2}sD(s)[\eta(s,\theta_0)-\tan(\pi s)]r^{-s-1}\,ds \tag{22}$$

$$-\frac{1}{2\pi i}\int_{B_r}[\psi_N(s,\theta_0)\bar{N}(s)+\psi_S(s,\theta_0)\bar{S}(s)]r^{-s-1}\,ds-p(r),\qquad 0<r<1.$$

4. SOLUTION OF THE DUAL INTEGRAL EQUATION

Single pairs of dual integral equations involving Bromwich integrals have been considered before, specifically in Ref. [1]. In a similar manner, introduce the following Mellin transform [6], such that eqn (21) is satisfied automatically,

$$D(s)=B(\tfrac{1}{2},s)\int_0^1 g(t)t^{s-\frac{1}{2}}\,dt \tag{23}$$

where $B(z,w)$ is the Beta function and $g(t)$ is a function to be determined. This form for $D(s)$ is seen to satisfy eqn (21) exactly [6]. Now note the identities, also from [6],

$$B(\tfrac{1}{2},s)\tan(\pi s)=B(\tfrac{1}{2},\tfrac{1}{2}-s) \tag{24}$$

$$\frac{1}{2\pi i}\int_{B_r}D(s)\tan(\pi s)r^{-s}\,ds=\int_0^r\frac{g(t)}{(r-t)^{1/2}}\,dt. \tag{25}$$

Thus, upon integrating eqn (22) with respect to r from 0 to r and with the use of eqns (23)–(25), the resulting equation is obtained,

$$\int_0^r\frac{g(t)}{(r-t)^{1/2}}\,dt=\frac{-1}{2\pi i}\int_{B_r}D(s)[\eta(s,\theta_0)-\tan(\pi s)]r^{-s}\,ds$$

$$-\frac{1}{2\pi i}\int_{B_r}2s^{-1}[\psi_N(s,\theta_0)\bar{N}(s)+\psi_S(s,\theta_0)\bar{S}(s)]r^{-s}\,ds+P(r)\qquad 0<r<1 \tag{26a}$$

$$P(r)=\int_0^r p(r)\,dr. \tag{26b}$$

Equation (26a) can be viewed as an Abel integral equation with the right hand side considered known. Employing the well known solution [7], eqn (23) and the relations from [6] and [8], respectively,

$$\frac{d}{dt}\int_0^t \frac{u^{-s}}{(t-u)^{1/2}}\,du = -st^{-s-1/2}B(\tfrac{1}{2}, -s) \tag{27}$$

$$B(\tfrac{1}{2}, 1) = \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(\frac{3}{2})} = 2 \tag{28}$$

$$sB(\tfrac{1}{2}, -s)B(\tfrac{1}{2}, s) = \frac{-\pi}{\tan(\pi s)}, \tag{29}$$

the following Fredholm integral equation can be written for $g^*(t) = t^{1/2}g(r)$.

$$\pi g^*(t) = \pi \int_0^1 g^*(u)K\left(\frac{u}{t}, \theta_0\right)\frac{du}{u} + F_c(t) + F_w(t, \theta_0), \qquad 0 < t < 1 \tag{30a}$$

$$K(\xi, \theta_0) = \frac{-1}{2\pi i}\int_{B_r} [\eta(s, \theta_0) - \tan(\pi s)]\frac{\xi^s\,ds}{\tan(\pi s)} \tag{30b}$$

$$F_c(t) = 2t^{1/2}\int_0^t \frac{p(u)\,du}{(t-u)^{1/2}} \tag{30c}$$

$$F_w(t, \theta_0) = \frac{1}{2\pi i}\int_{B_r} 2[\psi_N(s, \theta_0)\bar{N}(s) + \psi_S(s, \theta_0)\bar{S}(s)]B(\tfrac{1}{2}, -s)t^{-s}\,ds. \tag{30d}$$

A simple residue calculation shows that the stresses are $0(r^{-s^*-1})$ as $r \to 0^+$, where s^* is the first zero of $\Delta(s, \theta_0)$ to the left of zero, eqn (16c). It is easily shown that as θ_0 increases from 0 to π, s^* increases from $-\frac{3}{2}$ to $-\frac{1}{2}$. Thus, the parameter α given in eqn (13a) takes on values between $-\frac{1}{2}$ and $\frac{1}{2}$, and the Bromwich path is chosen to be $(-\frac{1}{4} - i\infty)$ to $(-\frac{1}{4} + i\infty)$ for all values of θ_0. With this choice, the complex line integrals, eqns (30b, d) can be written as real semi-infinite integrals, and can be evaluated numerically, using a standard Gaussian quadrature.

The following asymptotic behavior of $K(\xi, \theta_0)$ is obtained from a residue calculation,

$$K(\xi, \theta_0) = 0(\xi^{-s^*}) \qquad \text{as } \xi \to 0^+ \tag{31a}$$

$$= 0(\xi^{s^*}) \qquad \text{as } \xi \to \infty \tag{31b}$$

where s^* is defined as above.

If eqn (30a) is solved numerically for $g^*(t)$ and this result successively substituted into eqns (23), (18), (17), (16), and finally into eqns (6)–(10), the stress and displacement fields can be determined.

The quantities of physical interest are the normal tractions on $\theta = 0$, $r \in (1, \infty)$ which yield the stress intensity factor, and the crack opening displacement, $(v)_{\theta=0}$, $r \in [0, 1]$. Using the relation given in eqn (25) and substituting as indicated above,

$$\tau_{\theta\theta}(r, 0) = -\frac{1}{4}\int_0^1 \frac{g(t)}{(r-t)^{3/2}}\,dt + \int_0^1 g(t)\tilde{K}\left(\frac{t}{r}, \theta_0\right)t^{-3/2}\,dt + \tau_{\theta\theta}^{(w)}(r, \theta) \qquad 1 < r < \infty \tag{32a}$$

$$\tilde{K}(\xi, \theta_0) = \frac{1}{2\pi i}\int_{B_r} \tfrac{1}{2}s[\eta(s, \theta_0) - \tan(\pi s)]B(\tfrac{1}{2}, s)\xi^{(s+1)}\,ds \tag{32b}$$

where $\tau_{\theta\theta}^{(w)}(r, 0)$ is the first term on the right hand side of eqn (20a). The last two integrals in eqn (32a) are bounded as $r \to 1^+$ and only the first integral is singular at this point. Integrating by parts yields the result,

$$\tau_{\theta\theta}(r, 0) = \frac{-g(1)}{2(r-1)^{1/2}} + 0(1) \quad \text{as} \quad r \to 1^+. \tag{33}$$

In the usual manner, the crack tip stress intensity factor, K, is defined as

$$K = \lim_{r\to 1^+} [2(r-1)]^{1/2}\tau_{\theta\theta}(r,0) = -\tfrac{1}{2}(2)^{1/2}g(1). \quad (34)$$

In this dimensionless formulation, r is normalized with respect to the crack length; the only characteristic length in the problem is "a". An increase in this parameter corresponds to decreasing the crack length and *vice versa.*

Substituting successively again as indicated above and using the identity[6],

$$\begin{aligned}\frac{1}{2\pi i}\int_{B_r}\frac{\Gamma(s)}{\Gamma(\frac{1}{2}+s)}x^{-s}\,\mathrm{d}s &= [\Gamma(\tfrac{1}{2})(1-x)^{1/2}]^{-1} \qquad 0<x<1\\ &= 0 \qquad 1<x<\infty\end{aligned} \quad (35)$$

the crack opening displacement,

$$(2\mu)v(r,0) = -\int_r^1 \frac{g(t)}{(t-r)^{1/2}}\,\mathrm{d}t, \qquad 0<r<1, \quad (36)$$

is obtained.

5. NUMERICAL SOLUTION AND RESULTS

The integral equation eqn (30a) is solved by approximating the finite integral from $u=0$ to $u=1$ by an N*-pt.* Gaussian quadrature ($N=16$ was used for all computations). The collocation points, t_i, are chosen to be the same as the integration points, u_i, and are given by

$$u_i = t_i = \tfrac{1}{2}(x_i+1) \qquad i=1,\ldots,N \quad (37)$$

where the x_i are the N zeros of $P_N(x)$, the Nth Legendre polynomial, in the interval $(-1,1)$. The weights, w_i, for the quadrature formulas are given by

$$w_i = \frac{1}{(1-x_i^2)[P'_N(x_i)]^2} \qquad i=1,\ldots,N. \quad (38)$$

Thus, a linear system is obtained with the forcing vector $\mathbf{F}=\{F_c(t_i)+F_w(t_i)\}$ and the solution vector $\mathbf{G}=\{g^*(t_i)\}$, $i=1,\ldots,N$.

(a) *Constant normal load applied over crack and wedge faces*

One particular case which was considered for detailed numerical study was that of a constant normal load, with unit intensity, applied over the crack as well as on the wedge faces; i.e. $p(r)=N(r)=-1$ and $S(r)=0$. This gives

$$F_c(t) = -4t \quad (39a)$$

$$F_w(t) = \frac{-a}{\pi i}\int_{B_r}\frac{\psi_N(s,\theta_0)}{(s+1)}B(\tfrac{1}{2},-s)\left(\frac{a}{t}\right)^s\mathrm{d}s. \quad (39b)$$

As was the case with the kernel $K(\xi,\theta_0)$, the Bromwich integral in eqn (39b) is evaluated along the line $(-\frac{1}{4}-i\infty)\to(-\frac{1}{4}+i\infty)$.

Intensity factors were computed for angles θ_0 ranging from 60° to 179.95° and "a" ranging from 0.0 to 10.0. Note that $a=0.0$ corresponds to no load on the wedge faces. The results are given in Table 1. K_c represents the intensity factor due to loading on the crack faces and K_w represents the intensity factor due to loading on the wedge while the total stress intensity factor is $K=K_c+K_w$.

The results for several limiting cases were compared to previous results with considerable accuracy. The case $\theta_0=90°$ corresponds to the loaded half plane with a loaded edge crack. When $N(r)=S(r)=0.0$ we have the stress free half plane with a loaded edge crack. The latter problem was solved by Sneddon and Das[9] and the intensity factor arrived at by them was $K=1.1215$ compared to the presently computed value of $K=1.1207$. For $N(r)=1$ and $S(r)=0.0$ and

Table 1. The results for a cracked elastic wedge loaded normally on the faces from $r=0$ to $r=a$. The quantities shown are the stress intensity factors at the crack tip, $(r,\theta)=(1,0)$, for various $\frac{1}{2}$ wedge angles, θ_0. K_c is the intensity factor due to the crack load, $p(r)\equiv -1$ and K_w is that due to the wedge loading, $N(r)\equiv -1$. $K=K_c+K_w$

θ_0 \ a	0.0	2.0		4.0		6.0		8.0		10.0	
	K^*_c	K_w	K	K_w	K	K_w	K	K_w	K	K_w	K
60°	1.5928	-2.0906	-.4977	-2.2000	-.6073	-2.1894	-.5966	-2.1707	-.5799	-2.1453	-.5615
75°	1.3805	-1.2879	.0206	-1.4903	-.1818	-1.5423	-.2337	-1.5606	-.2521	-1.5676	-.2590
90°	1.1207	- .7544	.3662	- .9402	.1805	-1.0096	.1111	-1.0453	.0755	-1.0669	.0530
105°	1.0117	- .3257	.6860	- .3996	.6121	- .4136	.5981	- .4111	.6006	- .4027	.6090
120°	.9491	.0476	.9968	.1411	1.0902	.2386	1.1878	.3298	1.2789	.4138	1.3629
135°	.9167	.3418	1.2585	.5955	1.5162	.8181	1.7348	1.0099	1.9266	1.1824	2.0991
150°	.9027	.5346	1.4373	.9100	1.8127	1.2184	2.1211	1.4863	2.3890	1.7263	2.6290
165°	.8984	.6328	1.5312	1.0697	1.9682	1.4256	2.3240	1.7336	2.6320	2.0090	2.9075
179.95°	.8979	.6601	1.5580	1.1142	2.0121	1.4832	2.3811	1.8022	2.7001	2.0873	2.9853

*I.e. $K_w=0$, $K=K_c$. This is the same K_c for all values of a.

$\theta_0=90°$, it is possible to compute by elementary means[10] the value of the stresses along $y=0$ due to a uniform load from $y=-a$ to $y=a$ on the surface, $x=0$, of the half plane. In particular one can obtain the stress normal to $y=0$ ($\theta=0$ in our problem), $\tau_y(x)$, in the range $0\le x\le 1$. Then the corresponding loading function would be

$$F_w(t)=-2t^{1/2}\int_0^t \frac{\tau_y(u)\,\mathrm{d}u}{(t-u)^{1/2}}, \tag{40}$$

see eqn (30c). Comparison of values of $F_w(t)$ obtained from eqn (40) differed by only 1.5% from those obtained from eqn (39b) with $\theta_0=90°$. The special case of $S(r)=1.0$ and $N(r)=0.0$ was also computed and compared to the analytical solution of a half plane with symmetric shear loading. The agreement in this case was better than 0.01%.

The other appropriate limiting case is $\theta_0=180°$, or that of a normally loaded semi-infinite slit in an infinite medium ($N(r)=p(r)=1.0$, and $S(r)=0.0$). The solution to this problem is easily obtained in closed form using the method of Ref. [11], and is given by

$$K=\frac{2^{1/2}}{\pi}\int_0^{1+a} t^{-1/2}\,\mathrm{d}t=\frac{2}{\pi}[2(1+a)]^{1/2}, \tag{41}$$

for a uniform unit load applied from the crack tip to a distance of $(1+a)$ from the tip. In particular, for $a=10.0$, eqn (41) gives $K=2.9879$ compared to the presently computed value of $K=2.9853$ for $\theta_0=179.95°$.

The most striking feature of the results in Table 1 is the fact that there is a critical angle θ_c, such that when $\theta_0<\theta_c$, the wedge loads have a net compressive effect on the crack tip, exhibited as $K_w<0$. When $\theta_0>\theta_c$, however, the wedge loads enhance the tensile crack load by inducing a $K_w>0$. θ_c depends upon the parameter "a" and for all "a" considered, was between 105° and 120°. This phenomenon reflects itself in another critical angle, also dependent upon "a", which represents a zero net stress at the crack tip, i.e. $K_w=-K_c$. This critical angle was found to lie between 60° and 90° for the cases considered.

It is also of interest to note that the contour integral, $K(\xi,\theta_0)$, eqn (30b), can be approximated by two one-term residue approximations for small and large ξ respectively. $F_w(t)$ may be approximated, similarly by a one-term residue approximation for small t only. This approximation was carried out numerically for $\theta_0=120°$ and "a" = 10.0, using the critical value, $\xi_c=10.0$, as the boundary of approximation between small and large ξ and $K(\xi,\theta_0)$. The resulting intensity factors are $K_w=0.3999$, $K_c=0.8503$, $K=1.2502$, and should be compared to results in Table 1.

One can also obtain from these results the stress intensity factor corresponding to the loading

$$\begin{aligned} N(r)&=0 && 0<r<b\\ &=-1 && b<r<c\\ &=0 && c<r<\infty \end{aligned} \tag{42a–c}$$

$$S(r) = 0 \qquad 0 < r < \infty \tag{43}$$

by subtracting the stress intensity factor given in Table 1 for $a = 1$ from that given for $a = c$. In this manner the value of "b" can be found for a given "c", which yields a zero stress intensity factor.

(b) *Approximation to problem of a half plane with finite cracked notch*

It is of interest that an approximation for a half plane with a finite notch cracked at the apex, see Fig. 2, can be obtained from the present work when $c \gg 1$. The approximation is based on an analysis of the dependence of the intensity factor at the crack tip upon the halfnotch angle, $\psi = \pi - \theta_0$, in the unbounded medium, by varying "a" with θ_0 such that c remains constant. The uniform tensile load applied at infinity is equivalent to the loads $p(r) = -1.0$, $N(r) = -\cos^2(\theta_0)$, and $S(r) = -\sin(\theta_0) \times \cos(\theta_0)$, taking $\sigma = 1.0$. Intensity factors were calculated for various half-notch angles, ψ, with $c = 10.0$. The results are given in Table 2 and Fig. 3 (solid curve).

It is seen from these results for the infinite wedge, which are plotted as the solid curve in Fig. 3, that the stress intensity factor increases from the value given by eqn (41) with $a = 10$ to a maximum at about $\psi = 52°$ and then decreases rapidly. When the limit of 90° is reached, the curve properly approaches the value, $K_{90} = 1.1215$ for a half plane with a crack. The values of the stress intensity factor do not change appreciably from a constant value when the half-notch angle is less than 30°. Now for the half-plane problem with $a = 10$ and $\psi > 60°$ it is reasonable to assume that the influence of the free surface is relatively small and becomes smaller for larger angles; thus the solid curve should be adequate for the half-plane problem for $60° < \psi < 90°$. One can now approximate the effect of the free surface for smaller angles by the dashed curve shown in Fig. 3.

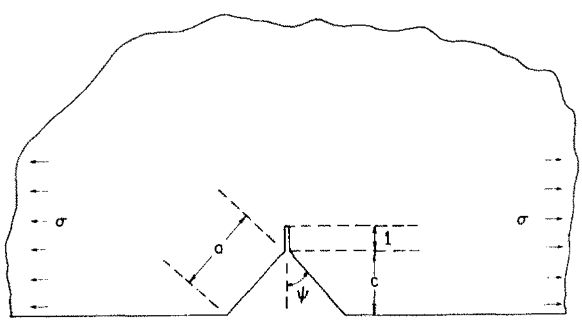

Fig. 2. Geometry and loading for a tensile specimen with a notch; $\theta_0 = \pi - \psi$, $c = a\cos(\psi)$.

Table 2. Results for approximation to the half-plane with a cracked notch. K_c is the intensity factor due to the crack load, $p(r) = 1$. K_N is that due to the normal load, $N(r) = -\cos^2(\psi)$. K_S is due to the shear load, $S(r) = \sin(\psi)\cos(\psi)$. $K = K_c + K_N + K_S$

ψ	K_c	K_N	K_S	K
1°	.8979	2.0866	.0009	2.9854
5°	.8979	2.0688	.0220	2.9888
10°	.8981	2.0123	.0898	3.0002
15°	.8984	1.9173	.2049	3.0207
22.5°	.8998	1.7059	.4640	3.0698
30°	.9027	1.4230	.8134	3.1391
37.5°	.9080	1.0921	1.2204	3.2205
45°	.9167	.7480	1.6311	3.2958
52.5°	.9299	.4333	1.9706	3.3339
60°	.9491	.1886	2.1513	3.2890
67.5°	.9758	.0391	2.0884	3.1034
75°	1.0117	-.0172	1.7215	2.7160
80°	1.0419	-.0180	1.2974	2.3213
85°	1.0779	-.0069	.7362	1.8072
89°	1.1116	-.0003	.1806	1.2919

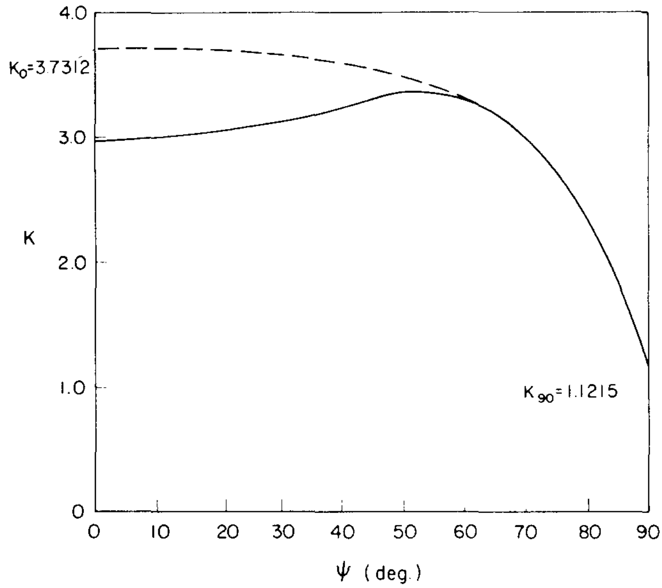

Fig. 3. K vs ψ for $c = 10.0$. Dashed line represents approximation to half plane with notch cracked at apex. Limiting values, K_0 and K_{90}, are given in text.

It will have the asymptotic value, $K_0 = 3.7312$ (for an edge crack of length 11, eqn (41)), which should remain relatively constant for $\psi < 30°$. Since a smooth dependence upon angle is expected, the connecting dashed curve shown for $30° < \psi < 60°$ should give a fair approximation to the actual stress intensity factor in this range.

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